

Least-squares spectral element preconditioners for fourth order elliptic problems

Akhlaq Husain^{a,*}, Arbaz Khan^{b,**}

^a*School of Engineering & Technology, BML Munjal University, Gurgaon-122413, Haryana, India*

^b*Interdisziplinäres Zentrum für Wissenschaftliches Rechnen (IWR), Ruprecht-Karls-Universität Heidelberg, 69120 Heidelberg, Germany*

Abstract

The goal of this paper is to propose preconditioners for the system of linear equations that arises from a discretization of fourth order elliptic problems using spectral element methods. These preconditioners are constructed using separation of variables and can be diagonalized and hence easy to invert. For second order elliptic problems this technique has proven to be very successful and performs better than other preconditioners. We show that these preconditioners are spectrally equivalent to the quadratic forms by which we approximate them. Numerical result for the biharmonic problem are presented to validate the theoretical estimates.

Keywords:

Fourth order problems, preconditioners, spectral element method, separation of variables, spectral equivalence, condition number.

1. Introduction

In this paper we investigate methods for preconditioning the system of linear equations that arises from a discretization of fourth order elliptic problems using spectral element methods. Preconditioners are usually constructed and analyzed with the goal of maintaining a well-conditioned system of equations as the number of unknowns W increases.

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded curvilinear domain having smooth boundary. We consider the model fourth order elliptic problem with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} \mathcal{L}u &= \Delta^2 u - \nabla \cdot (a \nabla u) + (b \cdot \nabla)u + cu = f \quad \text{in } \Omega, \\ u &= \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here, $\partial\Omega$ is the boundary of Ω , $\frac{\partial u}{\partial n}$ is the outer normal derivative on $\partial\Omega$, and $f \in L^2$. The coefficients b, c and the entries in the 2×2 matrix a are analytic. Moreover, a is symmetric, positive definite matrix. Problem (1.1) includes the classical plate bending problem in the theory

*This work was carried out during the second author's stay at the LNM Institute of Information Technology (LNMIIT), Jaipur as assistant professor

**Corresponding author

Email addresses: akhlaq.husain@bml.edu.in (Akhlaq Husain),
arbazkha@gmail.com, arbaz.khan@iwr.uni-heidelberg.de (Arbaz Khan)

of elasticity. This type of boundary value problems arise in structural mechanics, materials science and fluid flow. Such problems are also associated, for example, with the Cahn-Hilliard model for phase-separation phenomena [7].

Suppose that (1.1) is discretized using a spectral/finite element or finite difference method characterized by a mesh-size h . This yields an equation

$$Au^h = f. \quad (1.2)$$

in a finite-dimensional approximation space V_h . Here, A is usually symmetric and positive definite. An effective method for solving (1.2) consists of first preconditioning A and then using a convergent iterative method such as the preconditioned conjugate gradient method (PCGM). At every iteration, this method requires the evaluation of the matrix vector multiplications Ax and Br , where B is another symmetric, positive definite matrix, called *preconditioner*. The number of iterations required for the iterative method to converge depends on the condition number κ defined by

$$\kappa = \|BA\|_{V_h} \|(BA)^{-1}\|_{V_h} = \frac{\lambda_{\max}(BA)}{\lambda_{\min}(BA)}, \quad (1.3)$$

where $\|\cdot\|_{V_h}$ denotes the norm in V_h and $\lambda_{\max}, \lambda_{\min}$ are the largest, least eigenvalues respectively. It is well known that the number of iterations, required to achieve a given tolerance in the energy norm is proportional to $\sqrt{\kappa}$. The condition number and hence the computational cost increases rapidly as $h \rightarrow 0$ unless a suitable preconditioner is employed. Various preconditioners were developed and analyzed in [5, 6, 8] (as well as references cited therein) and shown to be extremely effective.

The preconditioners described in this paper are obtained in [12] for solving fourth order elliptic problems which are defined using a quadratic form which measures the H^4 norm of the spectral element function representation of the solution. These preconditioners are obtained in the same way as in [10, 15] by computing the residuals in the normal equations, but with homogeneous boundary data and the homogenous form of the partial differential equation. Hence, the algorithm is quite simple and easy to implement. We show that there exists a new diagonal preconditioner using separation of variables technique.

It is shown in [12] that the condition number of the preconditioned system grows like $O((\ln W)^4)$, where W denotes the polynomial degree and since we have mapped all elements in the domain Ω onto the master square $S = (-1, 1)^2$, therefore, we present the numerical results for a single reference element only. Unless otherwise stated, all the generic constants of approximation appearing in this paper are independent of N and depend on the shape regularity of Ω . Here, N denotes the number of elements in Ω . For computational simplicity, we assume that the degrees of the approximating polynomials are uniform in both directions. However, we can allow for non-uniform distribution of polynomial degree in each direction with a reduced number of degrees of freedom. This is usually done in the anisotropic case (e.g. in presence of singularities) where one uses lower order polynomials near singularities and higher order polynomials away from singularities to increase the effectiveness of the preconditioner.

The contents of this paper are now provided. In Section 2 preconditioners for elliptic problems are examined and numerical results are presented. In Section 3 we describe solution techniques for solving the system of linear equations arising from the spectral element discretization. Concluding remarks are provided in Section 4.

2. Preconditioners

Our construction of preconditioners is similar to that for second order elliptic and parabolic problems (see [8, 10]). We construct a preconditioner $\mathcal{B}(u)$ on each of the elements in Ω . We shall prove (as in [8]) that there is another quadratic form $\mathcal{C}(u)$ which is spectrally equivalent to $\mathcal{B}(u)$ and which can be easily diagonalized using the separation of variables. Then the matrix corresponding to the quadratic form $\mathcal{C}(u)$ will be easy to invert.

The preconditioner which needs to be examined corresponds to the quadratic form

$$\mathcal{B}(u) = \|u\|_{H^4(S)}^2, \quad (2.1)$$

where $u = u(\xi, \eta)$ is a polynomial of degree W in ξ and η separately. Let $u(\xi, \eta)$ be the spectral element function, defined on S , as

$$u(\xi, \eta) = \sum_{i=0}^W \sum_{j=0}^W a_{i,j} L_i(\xi) L_j(\eta). \quad (2.2)$$

Here, $L_i(\cdot)$ denotes the Legendre polynomial of degree i .

The quadratic form $\mathcal{B}(u)$ can be written as

$$\mathcal{B}(u) = \int_S \sum_{|\alpha| \leq 4} |D_{\xi, \eta}^\alpha u|^2 d\xi d\eta. \quad (2.3)$$

Define the quadratic form

$$\mathcal{C}(u) = \int_S (u_{\xi\xi\xi\xi}^2 + u_{\eta\eta\eta\eta}^2 + u_{\xi\xi\xi}^2 + u_{\eta\eta\eta}^2 + u_{\xi\xi}^2 + u_{\eta\eta}^2 + u_\xi^2 + u_\eta^2 + u^2) d\xi d\eta. \quad (2.4)$$

We now show that the quadratic form $\mathcal{C}(u)$ is spectrally equivalent to the quadratic form $\mathcal{B}(u)$, defined in (2.1) and can be diagonalized in the basis $\psi_{i,j}(\xi, \eta)$. Note that $\{\psi_{i,j}(\xi, \eta)\}_{i,j}$ is the tensor product of the polynomials $\phi_i(\xi)$ and $\phi_j(\eta)$.

To prove that $\mathcal{B}(u)$ and $\mathcal{C}(u)$ are spectrally equivalent we need to show that there is an extension $U(\xi, \eta)$ of $u(\xi, \eta)$ such that $U(\xi, \eta) \in H^4(\mathbb{R}^2)$ and satisfies the estimate

$$\int_{\mathbb{R}^2} (U_{\xi\xi\xi\xi}^2 + U_{\eta\eta\eta\eta}^2 + U_{\xi\xi}^2 + U_{\eta\eta}^2 + U^2) d\xi d\eta \leq K \int_S (u_{\xi\xi\xi\xi}^2 + u_{\eta\eta\eta\eta}^2 + u_{\xi\xi}^2 + u_{\eta\eta}^2 + u^2) d\xi d\eta.$$

Here, K is a constant which is independent of N .

To extend $u(\xi, \eta)$ defined on $(-1, 1) \times (-1, 1)$ the method of successive reflections is used. In the first step an extension $U_1(\xi, \eta)$ is obtained by reflecting $u(\xi, \eta)$ along the line $\eta = 1$. This construction is similar to that in Theorem 5.19 of [1].

For $\nu > 0$ define

$$U_1(\xi, 1 + \nu) = \sum_{l=1}^5 a_l u(\xi, 1 - l\nu) \Theta(\nu), \quad k = 0, 1, \dots, 4.$$

Here, $\Theta(\nu)$ is a C^∞ function such that

$$\Theta(\nu) = 1 \quad \text{for} \quad \nu \leq \frac{1}{6}, \quad \Theta(\nu) = 0 \quad \text{for} \quad \nu \geq \frac{1}{3}.$$

In addition, the coefficients a_i , $1 \leq i \leq 5$ are chosen to satisfy the 5×5 system of linear equations

$$\sum_{l=1}^5 (-l)^k a_l = 1, \quad k = 0, 1, \dots, 4.$$

Thus, the extension $U_1(\xi, \eta)$ of $u(\xi, \eta)$ can be written as

$$U_1(\xi, \eta) = \begin{cases} u(\xi, \eta), & -1 < \eta < 1 \\ \sum_{l=1}^5 a_l u(\xi, 1 - l(\eta - 1)) \Theta(\eta - 1), & \eta \geq 1. \end{cases} \quad (2.5)$$

Therefore, using (2.5) we can write

$$\begin{aligned} & \int_{-1}^{\infty} \int_{-1}^1 ((U_1)_{\xi\xi\xi\xi}^2 + (U_1)_{\eta\eta\eta\eta}^2 + (U_1)_{\xi\xi}^2 + (U_1)_{\eta\eta}^2 + (U_1)^2) \, d\xi d\eta. \\ &= \int_S (u_{\xi\xi\xi\xi}^2 + u_{\eta\eta\eta\eta}^2 + u_{\xi\xi}^2 + u_{\eta\eta}^2 + u^2) \, d\xi d\eta \\ &+ \int_1^{\infty} \int_{-1}^1 ((U_1)_{\xi\xi\xi\xi}^2 + (U_1)_{\eta\eta\eta\eta}^2 + (U_1)_{\xi\xi}^2 + (U_1)_{\eta\eta}^2 + (U_1)^2) \, d\xi d\eta \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \int_{-1}^{\infty} \int_{-1}^1 ((U_1)_{\xi\xi\xi\xi}^2 + (U_1)_{\eta\eta\eta\eta}^2 + (U_1)_{\xi\xi}^2 + (U_1)_{\eta\eta}^2 + (U_1)^2) \, d\xi d\eta. \\ & \leq K_1 \int_S (u_{\xi\xi\xi\xi}^2 + u_{\eta\eta\eta\eta}^2 + u_{\eta\eta}^2 + u_{\xi\xi}^2 + u_{\eta\eta}^2 + u_{\eta}^2 + u^2) \, d\xi d\eta. \end{aligned} \quad (2.6)$$

Here K_1 is a constant which is independent of W .

Applying Lemma 5.6 of [1], the following estimates hold

$$\int_{-1}^1 u_{\eta}^2(\xi, \eta) \, d\eta \leq C_1 \int_{-1}^1 (u_{\eta\eta}^2 + u^2) \, d\eta, \quad (2.7)$$

and

$$\int_{-1}^1 u_{\eta\eta\eta}^2(\xi, \eta) \, d\eta \leq C_2 \int_{-1}^1 (u_{\eta\eta\eta\eta}^2 + u^2) \, d\eta, \quad (2.8)$$

where C_1 and C_2 are constants.

Integrating (2.7), (2.8) with respect to ξ , we get

$$\int_S u_{\eta}^2(\xi, \eta) \, d\xi d\eta \leq C_1 \int_S (u_{\eta\eta}^2 + u^2) \, d\xi d\eta, \quad (2.9)$$

and

$$\int_S u_{\eta\eta\eta}^2(\xi, \eta) \, d\xi d\eta \leq C_2 \int_S (u_{\eta\eta\eta\eta}^2 + u^2) \, d\xi d\eta. \quad (2.10)$$

Combining (2.6), (2.7) and (2.8), imply the following estimate

$$\int_{-1}^{\infty} \int_{-1}^1 ((U_1)_{\xi\xi\xi\xi}^2 + (U_1)_{\eta\eta\eta\eta}^2 + (U_1)_{\xi\xi}^2 + (U_1)_{\eta\eta}^2 + (U_1)^2) d\xi d\eta \leq C \int_S (u_{\xi\xi\xi\xi}^2 + u_{\eta\eta\eta\eta}^2 + u_{\xi\xi}^2 + u_{\eta\eta}^2 + u^2) d\xi d\eta. \quad (2.11)$$

Here, C is a generic constant. We are now in a position to prove the lemma.

Lemma 2.1. *Let $u(\xi, \eta)$ be the polynomial as defined in (2.2). Then there is an extension $(Eu)(\xi, \eta) = U(\xi, \eta)$ of $u(\xi, \eta)$ such that $U(\xi, \eta) \in H^4(\mathbb{R}^2)$ and satisfies the following estimate*

$$\int_{\mathbb{R}^2} (U_{\xi\xi\xi\xi}^2 + U_{\eta\eta\eta\eta}^2 + U_{\xi\xi}^2 + U_{\eta\eta}^2 + U^2) d\xi d\eta \leq K \int_S (u_{\xi\xi\xi\xi}^2 + u_{\eta\eta\eta\eta}^2 + u_{\xi\xi}^2 + u_{\eta\eta}^2 + u^2) d\xi d\eta, \quad (2.12)$$

where K is a constant independent of W .

Proof. Assume that $U_1(\xi, \eta)$ be the extension of $u(\xi, \eta)$, defined on $(-1, 1) \times (-1, \infty)$, obtained by reflecting $u(\xi, \eta)$ about the line $\eta = 1$. Then

$$\begin{aligned} & \int_{-1}^{\infty} \int_{-1}^1 ((U_1)_{\xi\xi\xi\xi}^2 + (U_1)_{\eta\eta\eta\eta}^2 + (U_1)_{\xi\xi}^2 + (U_1)_{\eta\eta}^2 + (U_1)^2) d\xi d\eta \\ & \leq K_1 \int_S (u_{\xi\xi\xi\xi}^2 + u_{\eta\eta\eta\eta}^2 + u_{\xi\xi}^2 + u_{\eta\eta}^2 + u^2) d\xi d\eta. \end{aligned} \quad (2.13)$$

Let $U_2(\xi, \eta)$ to be the extension of $U_1(\xi, \eta)$, defined on $(-1, 1) \times (-\infty, \infty)$, obtained by reflecting $U_1(\xi, \eta)$ about the line $\eta = -1$ then we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-1}^1 ((U_2)_{\xi\xi\xi\xi}^2 + (U_2)_{\eta\eta\eta\eta}^2 + (U_2)_{\xi\xi}^2 + (U_2)_{\eta\eta}^2 + (U_2)^2) d\xi d\eta \\ & \leq K_2 \int_{-1}^{\infty} \int_{-1}^1 ((U_1)_{\xi\xi\xi\xi}^2 + (U_1)_{\eta\eta\eta\eta}^2 + (U_1)_{\xi\xi}^2 + (U_1)_{\eta\eta}^2 + (U_1)^2) d\xi d\eta. \end{aligned} \quad (2.14)$$

Let $U_3(\xi, \eta)$ to be the extension of $U_2(\xi, \eta)$, defined on $(-1, \infty) \times (-\infty, \infty)$, obtained by reflecting $U_2(\xi, \eta)$ about the line $\xi = 1$. Clearly,

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-1}^{\infty} ((U_3)_{\xi\xi\xi\xi}^2 + (U_3)_{\eta\eta\eta\eta}^2 + (U_3)_{\xi\xi}^2 + (U_3)_{\eta\eta}^2 + (U_3)^2) d\xi d\eta \\ & \leq K_3 \int_{-\infty}^{\infty} \int_{-1}^1 ((U_2)_{\xi\xi\xi\xi}^2 + (U_2)_{\eta\eta\eta\eta}^2 + (U_2)_{\xi\xi}^2 + (U_2)_{\eta\eta}^2 + (U_2)^2) d\xi d\eta. \end{aligned} \quad (2.15)$$

Finally, let $U(\xi, \eta)$ to be the extension of $U_3(\xi, \eta)$, defined on $(-\infty, \infty) \times (-\infty, \infty)$, obtained by reflecting $U_3(\xi, \eta)$ about the line $\xi = -1$. It follows that the estimate

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (U_{\xi\xi\xi\xi}^2 + U_{\eta\eta\eta\eta}^2 + U_{\xi\xi}^2 + U_{\eta\eta}^2 + U^2) d\xi d\eta \\ & \leq K_4 \int_{-\infty}^{\infty} \int_{-1}^{\infty} ((U_3)_{\xi\xi\xi\xi}^2 + (U_3)_{\eta\eta\eta\eta}^2 + (U_3)_{\xi\xi}^2 + (U_3)_{\eta\eta}^2 + (U_3)^2) d\xi d\eta. \end{aligned} \quad (2.16)$$

Combining (2.13), (2.14), (2.15) and (2.16) and choosing $K = K_1 K_2 K_3 K_4$, imply the final estimate (2.12). \square

Theorem 2.1. *The quadratic forms $\mathcal{B}(u)$ and $\mathcal{C}(u)$ are spectrally equivalent.*

Proof. Assume that $U(\xi, \eta)$ be the extension of $u(\xi, \eta)$ as defined in Lemma 2.1 with $U(\xi, \eta)|_S = u(\xi, \eta)$. Furthermore, $U(\xi, \eta) \in H^4(\mathbb{R}^2)$ and satisfies the following estimate

$$\begin{aligned} & \int_{\mathbb{R}^2} (U_{\xi\xi\xi\xi}^2 + U_{\eta\eta\eta\eta}^2 + U_{\xi\xi}^2 + U_{\eta\eta}^2 + U^2) d\xi d\eta \\ & \leq K \int_S (u_{\xi\xi\xi\xi}^2 + u_{\eta\eta\eta\eta}^2 + u_{\xi\xi}^2 + u_{\eta\eta}^2 + u^2) d\xi d\eta. \end{aligned}$$

Let $\widehat{U}(\alpha, \beta)$ be the Fourier transform of $U(\xi, \eta)$. Then

$$\int_{\mathbb{R}^2} U_{\xi\xi\eta\eta}^2 d\xi d\eta = \int_{\mathbb{R}^2} |\alpha^2 \beta^2|^2 |\widehat{U}(\alpha, \beta)|^2 d\alpha d\beta \leq \left(\int_{\mathbb{R}^2} |\alpha^4|^2 |\widehat{U}(\alpha, \beta)|^2 d\alpha d\beta \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\beta^4|^2 |\widehat{U}(\alpha, \beta)|^2 d\alpha d\beta \right)^{\frac{1}{2}}$$

using Cauchy-Schwarz inequality.

Applying inverse Fourier transform, implies the following estimate

$$\int_{\mathbb{R}^2} |U_{\xi\xi\eta\eta}|^2 d\xi d\eta \leq \left(\int_{\mathbb{R}^2} |U_{\xi\xi\xi\xi}|^2 d\xi d\eta \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |U_{\eta\eta\eta\eta}|^2 d\xi d\eta \right)^{\frac{1}{2}}.$$

Using AM-GM inequality, we obtain

$$\int_{\mathbb{R}^2} |U_{\xi\xi\eta\eta}|^2 d\xi d\eta \leq \frac{1}{2} \left(\int_{\mathbb{R}^2} |U_{\xi\xi\xi\xi}|^2 + |U_{\eta\eta\eta\eta}|^2 d\xi d\eta \right). \quad (2.17)$$

Therefore, we have

$$\int_S |u_{\xi\xi\eta\eta}|^2 d\xi d\eta \leq \frac{1}{2} \left(\int_{\mathbb{R}^2} |U_{\xi\xi\xi\xi}|^2 + |U_{\eta\eta\eta\eta}|^2 d\xi d\eta \right).$$

Inserting the result of Lemma 2.1, the following estimate holds

$$\int_S |u_{\xi\xi\eta\eta}|^2 d\xi d\eta \leq \frac{K}{2} \left(\int_S |u_{\xi\xi\xi\xi}|^2 + |u_{\eta\eta\eta\eta}|^2 + |u_{\xi\xi}|^2 + |u_{\eta\eta}|^2 + |u|^2 d\xi d\eta \right). \quad (2.18)$$

Now, we estimate the following term

$$\int_{\mathbb{R}^2} U_{\xi\xi\eta\eta}^2 d\xi d\eta = \int_{\mathbb{R}^2} |\alpha \beta^3|^2 |\widehat{U}(\alpha, \beta)|^2 d\alpha d\beta \leq \left(\int_{\mathbb{R}^2} |\alpha \beta|^4 |\widehat{U}(\alpha, \beta)|^2 d\alpha d\beta \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\beta^4|^2 |\widehat{U}(\alpha, \beta)|^2 d\alpha d\beta \right)^{\frac{1}{2}}.$$

using Cauchy-Schwarz inequality.

Applying inverse fourier transform, implies the following estimate

$$\int_{\mathbb{R}^2} |U_{\xi\eta\eta\eta}|^2 d\xi d\eta \leq \left(\int_{\mathbb{R}^2} |U_{\xi\xi\eta\eta}|^2 d\xi d\eta \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |U_{\eta\eta\eta\eta}|^2 d\xi d\eta \right)^{\frac{1}{2}}.$$

Using AM-GM inequality, we obtain

$$\int_{\mathbb{R}^2} |U_{\xi\eta\eta\eta}|^2 d\xi d\eta \leq \frac{1}{2} \left(\int_{\mathbb{R}^2} |U_{\xi\xi\eta\eta}|^2 + |U_{\eta\eta\eta\eta}|^2 d\xi d\eta \right).$$

Moreover, we obtain

$$\int_S |u_{\xi\eta\eta\eta}|^2 d\xi d\eta \leq \frac{1}{2} \left(\int_{\mathbb{R}^2} |U_{\xi\xi\eta\eta}|^2 + |U_{\eta\eta\eta\eta}|^2 d\xi d\eta \right). \quad (2.19)$$

Applying (2.17) and Lemma 2.1 in (2.19), implies

$$\int_S |u_{\xi\eta\eta\eta}|^2 d\xi d\eta \leq \frac{K}{2} \left(\int_S |u_{\xi\xi\xi\xi}|^2 + |u_{\eta\eta\eta\eta}|^2 + |u|^2 d\xi d\eta \right). \quad (2.20)$$

Similarly,

$$\int_{\mathbb{R}^2} U_{\xi\xi\xi\eta}^2 d\xi d\eta = \int_{\mathbb{R}^2} |\alpha^3 \beta|^2 |\widehat{U}(\alpha, \beta)|^2 d\alpha d\beta \leq \left(\int_{\mathbb{R}^2} |\alpha^4|^2 |\widehat{U}(\alpha, \beta)|^2 d\alpha d\beta \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\alpha \beta|^4 |\widehat{U}(\alpha, \beta)|^2 d\alpha d\beta \right)^{\frac{1}{2}}.$$

using Cauchy-Schwarz inequality.

Proceeding as above we obtain,

$$\int_S |u_{\xi\xi\xi\eta}|^2 d\xi d\eta \leq \frac{1}{2} \left(\int_{\mathbb{R}^2} (|U_{\xi\xi\eta\eta}|^2 + |U_{\eta\eta\eta\eta}|^2) d\xi d\eta \right). \quad (2.21)$$

Again, Applying (2.17) and Lemma 2.1 in (2.21), implies

$$\int_S |u_{\xi\xi\xi\eta}|^2 d\xi d\eta \leq \frac{K}{2} \int_S (|u_{\xi\xi\xi\xi}|^2 + |u_{\eta\eta\eta\eta}|^2 + |u|^2) d\xi d\eta. \quad (2.22)$$

In a similar way it follows that,

$$\int_S |u_{\xi\eta\eta}|^2 d\xi d\eta \leq \frac{K}{2} \int_S (|u_{\xi\xi}|^2 + |u_{\eta\eta\eta\eta}|^2 + |u|^2) d\xi d\eta, \quad (2.23)$$

$$\int_S |u_{\xi\xi\eta}|^2 d\xi d\eta \leq \frac{K}{2} \int_S (|u_{\xi\xi\xi\xi}|^2 + |u_{\eta\eta}|^2 + |u|^2) d\xi d\eta, \quad (2.24)$$

and

$$\int_S |u_{\xi\eta}|^2 d\xi d\eta \leq \frac{K}{2} \int_S (|u_{\xi\xi}|^2 + |u_{\eta\eta}|^2 + |u|^2) d\xi d\eta. \quad (2.25)$$

Here, K is a generic constant independent of N . Using (2.18-2.25), we conclude that

$$\begin{aligned} \frac{1}{M} \|u\|_{H^4(S)}^2 &\leq \int_S (|u_{\xi\xi\xi\xi}|^2 + |u_{\eta\eta\eta\eta}|^2 + |u_{\xi\xi\xi}|^2 + |u_{\eta\eta\eta}|^2 + |u_{\xi\xi}|^2 + |u_{\eta\eta}|^2 \\ &\quad + |u_\xi|^2 + |u_\eta|^2 + |u|^2) d\xi d\eta \leq \|u\|_{H^4(S)}^2. \end{aligned}$$

The constant M in the above is independent of N and this proves the Lemma. \square

We now show that the quadratic form $\mathcal{C}(u)$ defined in (2.4) as

$$\mathcal{C}(u) = \int_S (u_{\xi\xi\xi\xi}^2 + u_{\eta\eta\eta\eta}^2 + u_{\xi\xi\xi}^2 + u_{\eta\eta\eta}^2 + u_{\xi\xi}^2 + u_{\eta\eta}^2 + u_\xi^2 + u_\eta^2 + u^2) d\xi d\eta.$$

can be diagonalized in the basis $\{\psi_{i,j}\}_{i,j}$. Here, u is a polynomial in ξ and η as defined in (2.2). Let $\tilde{\mathcal{C}}(f, g)$ denote the bilinear form induced by the quadratic form $\mathcal{C}(u)$. Then

$$\begin{aligned} \tilde{\mathcal{C}}(f, g) &= \int_Q (f_{\xi\xi\xi\xi} g_{\xi\xi\xi\xi} + f_{\eta\eta\eta\eta} g_{\eta\eta\eta\eta} + f_{\xi\xi\xi} g_{\xi\xi\xi} + f_{\eta\eta\eta} g_{\eta\eta\eta} \\ &\quad + f_{\xi\xi} g_{\xi\xi} + f_{\eta\eta} g_{\eta\eta} + f_\xi g_\xi + f_\eta g_\eta + fg) d\xi d\eta. \end{aligned} \quad (2.26)$$

Let I denote the interval $(-1, 1)$ and

$$v(\xi) = \sum_{i=0}^W \beta_i L_i(\xi). \quad (2.27)$$

Define $b = (\beta_0, \beta_1, \dots, \beta_W)^T$. Next, the quadratic forms $\mathcal{E}(v)$ and $\mathcal{F}(v)$ are defined as

$$\mathcal{E}(v) = \int_I (v_{\xi\xi\xi\xi}^2 + v_{\xi\xi\xi}^2 + v_{\xi\xi}^2 + v_\xi^2) d\xi, \quad (2.28)$$

and

$$\mathcal{F}(v) = \int_I v^2 d\xi. \quad (2.29)$$

Apparently, there exist $(W+1) \times (W+1)$ matrices E and F such that

$$\mathcal{E}(v) = b^T E b, \quad (2.30)$$

and

$$\mathcal{F}(v) = b^T F b. \quad (2.31)$$

Here, the matrices E and F are symmetric and F is positive definite.

Furthermore, there exist $W + 1$ eigenvalues $0 \leq \mu_0 \leq \mu_1 \leq \dots \leq \mu_W$ and $W + 1$ eigenvectors b_0, b_1, \dots, b_W of the symmetric eigenvalue problem

$$(E - \mu F)b = 0. \quad (2.32)$$

Therefore, we have

$$(E - \mu_i F)b_i = 0.$$

Here the eigenvectors b_i are normalized. Hence the following relations hold

$$b_i^T F b_j = \delta_j^i. \quad (2.33a)$$

and

$$b_i^T E b_j = \mu_i \delta_j^i, \quad (2.33b)$$

Define $b_i = (b_{i,0}, b_{i,1}, \dots, b_{i,W})$. Next, the polynomial $\phi_i(\xi)$ is defined as

$$\phi_i(\xi) = \sum_{j=0}^W b_{i,j} L_j(\xi) \text{ for } 0 \leq i \leq W. \quad (2.34)$$

Now, we define the polynomial $\psi_{i,j}$ which is as follows:

$$\psi_{i,j}(\xi, \eta) = \phi_i(\xi) \phi_j(\eta), \quad (2.35)$$

for $0 \leq i \leq W, 0 \leq j \leq W$.

Let $\tilde{\mathcal{E}}(f, g)$ and $\tilde{\mathcal{F}}(f, g)$ denote the corresponding bilinear forms induced by $\mathcal{E}(v)$ and $\mathcal{F}(v)$. Then

$$\tilde{\mathcal{E}}(f, g) = \int_I (f_{\xi\xi\xi\xi} g_{\xi\xi\xi\xi} + f_{\xi\xi\xi} g_{\xi\xi\xi} + f_{\xi\xi} g_{\xi\xi} + f_{\xi} g_{\xi}) d\xi, \quad (2.36a)$$

and

$$\tilde{\mathcal{F}}(f, g) = \int_I f g d\xi, \quad (2.36b)$$

where $f(\xi)$ and $g(\xi)$ are polynomials of degree W in ξ .

Furthermore, the relation (2.33a) and (2.33b) become

$$\tilde{\mathcal{F}}(\phi_i, \phi_j) = \int_I \phi_i(\xi) \phi_j(\xi) d\xi = \delta_j^i. \quad (2.37a)$$

and

$$\tilde{\mathcal{E}}(\phi_i, \phi_j) = \int_I ((\phi_i)_{\xi\xi\xi\xi} (\phi_j)_{\xi\xi\xi\xi} + (\phi_i)_{\xi\xi\xi} (\phi_j)_{\xi\xi\xi} + (\phi_i)_{\xi\xi} (\phi_j)_{\xi\xi} + (\phi_i)_{\xi} (\phi_j)_{\xi}) d\xi = \mu_i \delta_j^i. \quad (2.37b)$$

Combining (2.35), (2.37) and (2.26), it is easy to show that

$$\tilde{\mathcal{C}}(\psi_{i,j}, \psi_{k,l}) = (\mu_i + \mu_j + 1) \delta_k^i \delta_l^j = \mu_{i,j} \delta_k^i \delta_l^j.$$

Clearly, the eigenvectors of the quadratic form $\mathcal{C}(u)$ are $\{\psi_{i,j}\}_{i,j}$ and the eigenvalues are $\{\mu_{i,j}\}_{i,j}$ given by the relation

$$\mu_{i,j} = \mu_i + \mu_j + 1. \quad (2.38)$$

Moreover, the quadratic form $\mathcal{C}(u)$ can be diagonalized in the basis $\{\psi_{i,j}\}_{i,j}$ and consequently the matrix corresponding to $\mathcal{C}(u)$ is easy to invert.

Let κ denotes the condition number of the preconditioned system obtained by using the quadratic form $\mathcal{C}(u)$ as a preconditioner for the quadratic form $\mathcal{B}(u)$. Then the values of κ as a function of W are shown in Table 1.

W	$\log_{10}(\kappa)$
4	2.029731
8	2.114018
12	2.163258
16	2.191361
20	2.209824
24	2.223069
28	2.234161
32	2.239001

Table 1: Condition number κ as a function of W

3. Solution Techniques

Let

$$u(\xi, \eta) = \sum_{i=0}^W \sum_{j=0}^W \beta_{i,j} L_i(\xi) L_j(\eta) ,$$

and β denotes the column vector whose components are $\beta_{i,j}$ arranged in lexicographic order. Then there is a $(W+1)^2 \times (W+1)^2$ matrix C such that

$$\mathcal{C}(u) = \beta^T C \beta .$$

We now show (as in [8]) how to solve the system of equations

$$C\beta = \rho.$$

Define a polynomial $r(\xi, \eta)$ corresponding to the vector ρ is given by

$$r(\xi, \eta) = \sum_{i=0}^W \sum_{j=0}^W \rho_{i,j} L_i(\xi) L_j(\eta).$$

Here, the column vector ρ is obtained by arranging the elements $\rho_{i,j}$ in lexicographic order. Now by (2.34)

$$\phi_i(\xi) = \sum_{j=0}^W b_{i,j} L_j(\xi) \quad \text{for } 0 \leq i \leq W.$$

Inverting the above relation we have

$$L_i(\xi) = \sum_{j=0}^W \tilde{b}_{i,j} \phi_j(\xi). \tag{3.1}$$

Using (3.1), we may write

$$r(\xi, \eta) = \sum_{i=0}^W \sum_{j=0}^W \tilde{\rho}_{i,j} \phi_i(\xi) \phi_j(\eta).$$

Next, we define the polynomial $g(\xi, \eta)$ as

$$\begin{aligned} g(\xi, \eta) &= \sum_{i=0}^W \sum_{j=0}^W \frac{\tilde{\rho}_{i,j}}{\mu_{i,j}} \phi_i(\xi) \phi_j(\eta) , \\ &= \sum_{i=0}^W \sum_{j=0}^W \nu_{i,j} \phi_i(\xi) \phi_j(\eta). \end{aligned}$$

Here, $\nu_{i,j} = \frac{\tilde{\rho}_{i,j}}{\mu_{i,j}}$. Now

$$g(\xi, \eta) = \sum_{i=0}^W \sum_{j=0}^W \beta_{i,j} L_i(\xi) L_j(\eta).$$

We can obtain $\{\beta_{i,j}\}_{i,j}$ from $\{\nu_{i,j}\}_{i,j}$ using relation (2.34). Clearly, we can solve the system of equations (3.1) in $O(N^3)$ operations.

Let $u(\xi, \eta)$ be a polynomial as defined in (2.2) which vanishes at the vertices of the square $S = (-1, 1) \times (-1, 1)$.

Let

$$\begin{aligned} V_1(\xi) &= \frac{(1 - \xi)}{2} = \frac{L_0(\xi) - L_1(\xi)}{2}, \\ V_2(\xi) &= \frac{(1 + \xi)}{2} = \frac{L_0(\xi) + L_1(\xi)}{2}, \\ V_i(\xi) &= \sqrt{\frac{2i-3}{2}} \int_{-1}^{\xi} L_{i-2}(s) ds = \frac{1}{2\sqrt{(2i-3)}} (L_{i-1}(\xi) - L_{i-3}(\xi)) \quad \text{for } 3 \leq i \leq W+1, \end{aligned} \quad (3.2)$$

denote the hierarchic shape functions as defined in [14]. Then

$$V(\pm 1) = 0 \quad \text{for } 3 \leq i \leq W+1,$$

and $V_1(\xi)$ vanishes at $\xi = 1$. Moreover, $V_2(\xi)$ vanishes at $\xi = -1$. Let

$$\omega(\xi) = \sum_{i=3}^{W+1} \gamma_i V_i(\xi),$$

and $\mathcal{E}(\omega)$ and $\mathcal{F}(\omega)$ be the quadratic forms defined in (2.28) and (2.29). Clearly, there exist $W-1 \times W-1$ matrices \tilde{E} and \tilde{F} such that

$$\mathcal{E} = c^T \tilde{E} c,$$

and

$$\mathcal{F} = c^T \tilde{F} c.$$

Here, c denotes the vector

$$c = [\gamma_3, \gamma_4, \dots, \gamma_{W+1}]^T.$$

Moreover, the matrices \tilde{E} and \tilde{F} are symmetric and \tilde{F} is positive definite. Hence, there exist $W-1$ eigenvalues

$$0 \leq \nu_3 \leq \nu_4 \leq \dots \leq \nu_{W+1},$$

of the symmetric eigenvalue problem

$$(\tilde{E} - \nu \tilde{F})c = 0. \quad (3.3)$$

Let c_i be the eigenvector corresponding to the eigenvalue ν_i . Then

$$(\tilde{E} - \nu_i \tilde{F})c_i = 0. \quad (3.4)$$

Moreover the eigenvectors c_i are normalized so that

$$c_i^T \tilde{F} c_j = \delta_j^i. \quad (3.5)$$

In addition the relations

$$c_i^T \tilde{E} c_j = \nu_i \delta_j^i, \quad (3.6)$$

hold. Here $c_i = [c_{i,3}, c_{i,4}, \dots, c_{i,W+1}]^T$.

We now define the polynomials

$$h_i(\xi) = \sum_{j=3}^{W+1} c_{i,j} V_j(\xi) \quad \text{for } 3 \leq i \leq W+1. \quad (3.7)$$

Then

$$\int_{-1}^1 h_i h_j d\xi = \delta_j^i, \quad (3.8a)$$

and

$$\int_{-1}^1 ((h_i)_{\xi\xi\xi\xi}(h_j)_{\xi\xi\xi\xi} + (h_i)_{\xi\xi\xi}(h_j)_{\xi\xi\xi} + (h_i)_{\xi\xi}(h_j)_{\xi\xi} + (h_i)_{\xi}(h_j)_{\xi}) d\xi = \nu_i \delta_j^i. \quad (3.8b)$$

Now consider the bilinear form

$$\tilde{C}(f, g) = \int_S (f_{\xi\xi\xi\xi} g_{\xi\xi\xi\xi} + f_{\eta\eta\eta\eta} g_{\eta\eta\eta\eta} + f_{\xi\xi\xi} g_{\xi\xi\xi} + f_{\eta\eta\eta} g_{\eta\eta\eta} + f_{\xi\xi} g_{\xi\xi} + f_{\eta\eta} g_{\eta\eta} + f_{\xi} g_{\xi} + f_{\eta} g_{\eta}) d\xi d\eta,$$

defined in (2.26).

Let $P_{i,j}$ denote the polynomial

$$P_{i,j}(\xi, \eta) = h_i(\xi) h_j(\eta), \quad (3.9)$$

for $3 \leq i, j \leq W+1$. Then, using relations (3.8) it is easy to show that

$$\tilde{C}(P_{i,j}, P_{k,l}) = (\nu_i + \nu_j + 1) \delta_j^i \delta_l^k. \quad (3.10)$$

Now, if $u(\xi, \eta)$ is a polynomial as defined in (2.2) which vanishes at the vertices of the square S then it has the representation

$$\begin{aligned} u(\xi, \eta) = & \sum_{i=3}^{W+1} \sum_{j=3}^{W+1} o_{i,j} h_i(\xi) h_j(\eta) + \sum_{i=3}^{W+1} e_i h_i(\xi) V_1(\eta) + \sum_{i=3}^{W+1} f_i h_i(\xi) V_2(\eta) \\ & + \sum_{j=3}^{W+1} g_j V_1(\xi) h_j(\eta) + \sum_{j=3}^{W+1} h_j V_2(\xi) h_j(\eta). \end{aligned} \quad (3.11)$$

Or, we may write

$$u(\xi, \eta) = \sum_{i=3}^{W+1} \sum_{j=3}^{W+1} o_{i,j} P_{i,j}(\xi, \eta) + \sum_{i=1}^{4W-4} q_i R_i(\xi, \eta). \quad (3.12)$$

Here $\{R(\xi, \eta)\}_{i=1, 4W-4}$ denote the polynomials $\{h_i(\xi)V_1(\eta)\}_{i=3, \dots, W+1}$, $\{h_i(\xi)V_2(\eta)\}_{i=3, \dots, W+1}$, $\{V_1(\xi)h_j(\eta)\}_{j=3, \dots, W+1}$ and $\{V_2(\xi)h_j(\eta)\}_{j=3, \dots, W+1}$.

Let A denote the matrix of the bilinear form $\tilde{\mathcal{C}}(f, g)$ in the basis consisting of $\{P_{i,j}(\xi, \eta)\}_{i,j}$, $\{R_i(\xi, \eta)\}_i$. Then

$$A = \begin{bmatrix} D & E \\ E^T & F \end{bmatrix}.$$

Here D is a $(W-1)^2 \times (W-1)^2$ matrix and E is a $(W-1)^2 \times (4W-4)$ matrix. Moreover F is a $(4W-4) \times (4W-4)$ matrix.

Let o denote the vector whose components are $o_{i,j}$ arranged in lexicographic order and q the vector whose components are q_i . Here $\{o_{i,j}\}_{i,j}$ and $\{q_i\}_i$ are as in (3.12). Let p denote the vector

$$p = \begin{bmatrix} o \\ q \end{bmatrix},$$

and z denote the vector

$$z = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We now wish to solve the system of equations

$$Ap = z. \quad (3.13)$$

Define the Schur complement S of the system of equations (3.13) as

$$S = F - E^T D^{-1} E.$$

Then S is a $(4W-4) \times (4W-4)$ matrix.

To obtain the solution p of (3.13) we first solve

$$Sq = y - E^T D^{-1} x. \quad (3.14)$$

Next, we compute

$$o = D^{-1}(x - Eq). \quad (3.15)$$

It is easy to see that the system of equations (3.13) can be solved in $O(W^3)$ operations.

4. Conclusions

Preconditioners for fourth order elliptic using least-squares spectral element methods have been presented in this paper. Numerical results presented in Table 1 demonstrate the effectiveness of these preconditioners. We remark that the results and the method presented in this paper can be easily extended with obvious modifications to three-dimensional models and more general differential operators (e.g., variable coefficients, higher order, multidimensional differential operator).

References

- [1] R. A. Adams (1975): Sobolev Spaces; Academic Press, New York.
- [2] I. Babuška and B. Guo (1988): Regularity of the solutions of elliptic problems with piecewise analytic data. Part I. Boundary value problems for linear elliptic equation of second order; *SIAM J. Math. Anal.*, Vol. 19, No. 1, 172.
- [3] I. Babuška and B. Guo (1988): The $h - p$ version of the finite element method on domains with curved boundaries; *SIAM J. Num. Anal.*, Vol. 25, 837.
- [4] I. Babuška and B. Guo (1986): The $h - p$ Version of the finite element method, Part I: The basic approximation results; *Comp. Mech.*, 1, 21-41.
- [5] E. Bänsch, P. Morin and R. H. Nochetto (2011): Preconditioning a class of fourth order problems by operator splitting, *Numer. Math.*, 118, 197-228.
- [6] J. H. Bramble and J. E. Pasciak (1984): Preconditioned iterative methods for nonselfadjoint or indefinite elliptic boundary value problems, in *Unification of Finite Element Methods*, H. Kardestuncer, ed., North-Holland, Amsterdam.
- [7] J. W. Cahn, J. E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.* 28 (1958) 258-267.
- [8] P. Dutt, P. Biswas and G.N. Raju (2008): Preconditioners for spectral element methods for elliptic and parabolic problems; *Journal of Computational and Applied Mathematics*, Vol. 215, Issue 1, 152-166.
- [9] P. Dutt, P. Biswas and S. Ghorai (2007): Spectral element methods for parabolic problems, *J. Comput. Appl. Math.*, 203, 461-486.
- [10] P.K. Dutt, N. Kishore Kumar and C.S. Upadhyay (2007): Non-conforming $h - p$ spectral element methods for elliptic problems; *Proc. Indian Acad. Sci. (Math. Sci.)*, Vol. 117, No. 1, 109-145.
- [11] G. Karniadakis and J. Sherwin (1999): Spectral/ h - p Element Methods for CFD; *Oxford University Press*.
- [12] A. Khan, A. Husain, Least-squares spectral element methods for fourth order elliptic problems, submitted for publication.
- [13] D. Pathria and G.E. Karniadakis (1995): Spectral element methods for elliptic problems in non-smooth domains, *Journal of Computational Physics*, 122, 83-95.
- [14] C. Schwab (1998): p - and hp Finite Element Methods, *Clarendon Press*, Oxford, 1998.
- [15] S.K. Tomar (2006): $h - p$ spectral element methods for elliptic problems over non-smooth domains using parallel computers; *Computing*, 78, 117-143.